

Continuum model for radial interface growth

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A stochastic partial differential equation along the lines of the Kardar-Parisi-Zhang equation is introduced for the evolution of a growing interface in a radial geometry. Regular polygon solutions as well as radially symmetric solutions are identified in the deterministic limit. The polygon solutions, of relevance to on-lattice Eden growth from a seed in the zero-noise limit, are unstable in the continuum in favour of the symmetric solutions. The asymptotic surface width scaling for stochastic radial interface growth is investigated through numerical simulations and found to be characterized by the same scaling exponent as that for stochastic growth on a substrate.

1 Introduction

The Kardar-Parisi-Zhang (KPZ) equation [1]

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2} \left(\frac{\partial h}{\partial x} \right)^2 + \eta(x, t) \quad (1)$$

is the simplest nonlinear stochastic evolution equation for a growing interface. In this equation $h(x, t)$ denotes the height of an interface at substrate position x and time t ; $\eta(x, t)$ denotes a white noise term and ν and λ are growth parameters related to surface tension and lateral growth respectively. Among its applications the KPZ equation models vapor deposition at large length scales, Eden growth on a substrate, directed polymers, and one-dimensional turbulence. In the case where $\lambda = 0$ the KPZ equation reduces to the linear Edwards-Wilkinson (EW) equation [2] for random deposition. For reviews of

the extensive literature on the KPZ, EW and related equations and their applicability to evolving interfaces in a wide range of physical phenomena, we refer the reader to Refs. [3–6].

In this paper we are motivated by one of the above applications, Eden growth, to obtain an analogous equation to the KPZ equation in the case when the growth is radial from a seed, rather than from the usual substrate. We derive this equation in Section 3, after recalling the derivation of the KPZ equation in Section 2. In Section 4 we perform linear stability analysis for deterministic radially symmetric solutions. Numerical solutions are reported in Section 5, along with the exponent β governing the width of the interface. We conclude with a summary.

2 KPZ equation for growth on a substrate

The KPZ equation comprises three terms; surface tension, lateral growth and noise. The surface tension acts to smooth out the interface by rounding off bumps and filling in hollows. It is assumed that the time rate of change in the local height function due to surface tension is proportional to the surface curvature

$$\kappa = \frac{\frac{\partial^2 h}{\partial x^2}}{\left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right]^{\frac{3}{2}}}. \quad (2)$$

The basis of the lateral growth contribution in the KPZ equation is the assumption that all points on the growing interface move with uniform speed v in a direction normal to the interface. The geometry underlying lateral growth in the KPZ equation is shown in fig. 1. The lateral growth contribution to the time rate of change of the height function is found by projecting the outwards normal growth in a direction orthogonal to the substrate. Ignoring overhangs, the result which follows from the Pythagorean theorem (fig. 1) is

$$v \left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right]^{\frac{1}{2}}. \quad (3)$$

The noise term in the KPZ equation is assumed to be uncorrelated white noise;

$$\langle \eta(x, t) \rangle = 0 \quad (4)$$

$$\langle \eta(x, t) \eta(x', t') \rangle = 2D \delta(x - x') (t - t') \quad (5)$$

where D is a surface diffusion constant.

The reduction of

$$\frac{\partial h}{\partial t} = \nu \frac{\frac{\partial^2 h}{\partial x^2}}{\left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right]^{\frac{3}{2}}} + v \left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right]^{\frac{1}{2}} + \eta(x, t) \quad (6)$$

to the KPZ equation, Eq. (1), is achieved by making a small gradient approximation $\frac{\partial h}{\partial x} \ll 1$ and transforming to a co-moving frame.

3 Continuum equation for radial growth from a seed

We now consider surface tension, lateral growth and noise in a radial geometry. In this case the interface is characterized by its radial position $R(\theta, t)$ at angle θ . We assume that $R(\theta, t)$ is single valued so that there are no ‘overhangs’ along a direction of constant θ . The noise term is now a function of θ rather than x but otherwise it is identical to Eqs. (4), (5). The curvature of the surface characterized by $R(\theta, t)$ is given by

$$\frac{R^2 + 2\left(\frac{\partial R}{\partial \theta}\right)^2 - R\frac{\partial^2 R}{\partial \theta^2}}{\left[R^2 + \left(\frac{\partial R}{\partial \theta}\right)^2\right]^{\frac{3}{2}}}. \quad (7)$$

The geometry for lateral growth in a radial geometry is shown in fig. 2. Simple trigonometric manipulations lead to the lateral growth term

$$\frac{v}{R} \left[R^2 + \left(\frac{\partial R}{\partial \theta}\right)^2\right]^{\frac{1}{2}}. \quad (8)$$

Combining the radial noise term with surface tension proportional to the surface curvature, Eq. (7), and the lateral growth given by Eq. (8) then gives the continuum model for radial growth of an interface:

$$\frac{\partial R}{\partial t} = \frac{v}{R} \left[R^2 + \left(\frac{\partial R}{\partial \theta}\right)^2\right]^{\frac{1}{2}} - \nu \frac{R^2 + 2\left(\frac{\partial R}{\partial \theta}\right)^2 - R\frac{\partial^2 R}{\partial \theta^2}}{\left[R^2 + \left(\frac{\partial R}{\partial \theta}\right)^2\right]^{\frac{3}{2}}} + \eta(\theta, t). \quad (9)$$

The deterministic version of Eq. (9) has radially symmetric solutions $R(\theta, t) = f(t)$ where

$$\frac{df}{dt} = \frac{\nu}{f} + v. \quad (10)$$

In the absence of surface tension the radially symmetric solutions have the form $R(t) = vt + C$ whereas with the inclusion of surface tension $R(t)$ satisfies the transcendental equation

$$\nu \log(\nu + vR(t)) - vR(t) + v^2t = C. \quad (11)$$

Another class of deterministic ‘solutions’ are polygons defined by straight line segments

$$R(\theta, t) = \frac{vt}{\cos(\theta - c)} \quad (12)$$

where the parameter c is fixed between the vertices of the polygon. These solutions have zero curvature along the faces however they break down at the vertices where they are not differentiable. A special class are the regular n sided polygons

$$R_j(\theta, t) = \frac{vt}{\cos(\theta - \frac{\pi}{n} - \frac{2j\pi}{n})}, \quad \frac{2\pi j}{n} \leq \theta \leq \frac{2\pi(j+1)}{n} \quad (13)$$

with $j = 0, \dots, n-1$. We mention these ‘solutions’ here because exact regular polygons are obtained in on-lattice zero-noise simulations of the Eden A model from a seed, where on the square lattice, $n = 4$ and on the triangular and honeycomb lattices $n = 6$.

4 Linear stability analysis of deterministic solutions

The linear stability of solutions \bar{R} to the continuum model for radial growth can be investigated by substituting the perturbed solution $R = \bar{R} + \rho$ into the growth equation, Eq. (9), and retaining terms to first order in the perturbation ρ . This results in a linear partial differential equation to solve for the perturbation. In the special case where $\bar{R} = \bar{R}(t)$ are the radially symmetric solutions the perturbation satisfies

$$\frac{\partial \rho}{\partial t} = \frac{\nu}{\bar{R}(t)^2} \rho + \frac{\nu}{\bar{R}(t)^2} \frac{\partial^2 \rho}{\partial \theta^2}. \quad (14)$$

Assuming a separable solution $\rho = T(t)H(\theta)$ where $H(\theta) = H(\theta + 2\pi)$ now results in

$$\rho(\theta, t) = \frac{a_0}{2}e^{\nu\omega(t)} + \sum_{m=1}^{\infty} (a_m \cos(m\theta) + b_m \sin(m\theta))e^{(1-m^2)\nu\omega(t)} \quad (15)$$

where

$$\omega(t) = \int \frac{1}{R^2(t)} dt \quad (16)$$

is strictly positive. The coefficients a_m, b_m are determined by the initial conditions $\rho(\theta, t_0)$. In particular,

$$a_m = \frac{1}{\pi e^{(1-m^2)\nu\omega(t_0)}} \int_{-\pi}^{\pi} \rho(\theta, t_0) \cos(m\theta) d\theta. \quad (17)$$

$$b_m = \frac{1}{\pi e^{(1-m^2)\nu\omega(t_0)}} \int_{-\pi}^{\pi} \rho(\theta, t_0) \sin(m\theta) d\theta. \quad (18)$$

Hence if

$$\int_{-\pi}^{\pi} \rho(\theta, t_0) d\theta = \int_{-\pi}^{\pi} \rho(\theta, t_0) \cos(\theta) d\theta = \int_{-\pi}^{\pi} \rho(\theta, t_0) \sin(\theta) d\theta = 0 \quad (19)$$

then the perturbation will decay in time and the radially symmetric solutions will be asymptotically stable.

We can use the above analysis to investigate the linear stability of the regular polygon solutions. The idea is to consider an initial regular polygon profile as an initially perturbed radially symmetric solution $\bar{R}(t)$. In this case the perturbation initially satisfies

$$\rho(t_0) = R_j(\theta, t_0) - \bar{R}(t_0) \quad (20)$$

where $R_j(\theta, t_0)$ is defined in Eq. (13). It is a simple exercise to verify that the conditions in Eq. (19) are met in this case and thus the starting regular polygon will relax to the radially symmetric solution. This is a direct consequence of surface tension suppressing growth at the vertices. The persistence of regular polygon solutions in zero noise Eden growth is due to a balance between this surface tension and the underlying lattice anisotropy.

5 Numerical investigations

In general the stability of arbitrary profiles can be investigated directly by numerically integrating the continuum equation for radial interface growth with the given profile as an initial condition. In the numerical studies reported below we have employed the simple discretizations

$$R(\theta, t) = R(i\Delta\theta, j\Delta t) = R_{i,j} \quad (21)$$

$$\eta(\theta, t) = \eta(i\Delta\theta, j\Delta t) = \eta_{i,j} \quad (22)$$

$$\frac{\partial R}{\partial t} = \frac{R_{i,j+1} - R_{i,j}}{\Delta t} \quad (23)$$

$$\frac{\partial R}{\partial \theta} = \frac{R_{i+1,j} - R_{i-1,j}}{2\Delta\theta} \quad (24)$$

$$\frac{\partial^2 R}{\partial \theta^2} = \frac{R_{i+1,j} + R_{i-1,j} - 2R_{i,j}}{\Delta\theta^2}. \quad (25)$$

Fig. 3 shows time snapshots (at intervals of 100 time units) of the growing interface starting from a diamond in the deterministic limit, $\eta = 0$, with growth velocity $v = 0.1$ for the two cases; (a) $\nu = 0$ (solid line) and (b) $\nu = 0.1$ (dashed line). In this simulation $\Delta t = 1/20$ and $\Delta\theta = 2\pi/500$. In each case the diamond profile is smoothed out and approaches a circular profile in agreement with the predictions of the linear stability analysis above. In the integration with surface tension this approach to the circular profile is faster.

Fig. 4 shows time (number of iterations) snapshots of the growing interface starting from a circle in a stochastic integration with η a random number in the range $[-0.008, 0.008]$ and physical parameters $v = 1.0, \nu = 0.1$. In this simulation $\Delta t = 1/1000$ and $\Delta\theta = 2\pi/100$. The growing interface in Fig. 4 is similar in appearance to the growing interface in Eden growth [7].

Fig. 5 shows a log-log plot of the surface width versus time (number of iterations), averaged over five stochastic integrations, with parameters as in Fig. 4. Also shown on this plot is a straight line of best fit with slope $\beta = 1/3$. This suggests that the continuum equation, Eq. (9), for radial interface growth is in the same universality class as the KPZ equation.

6 Summary

We have derived the stochastic partial differential equation, Eq. (9), for the evolution of a growing interface in a radial geometry. The equation has regular polygon solutions as well as radially symmetric solutions in the deterministic

limit. Linear stability analysis reveals that the regular polygon solutions are unstable in favour of the radially symmetric solutions. The numerical solution of the fully stochastic equation indicates that the width of the interface scales with the same exponent, $\beta = 1/3$, as the KPZ equation. Presumably this result can be obtained exactly and thus provide explicit confirmation that both continuum models of interface growth lie in the same universality class.

Acknowledgement

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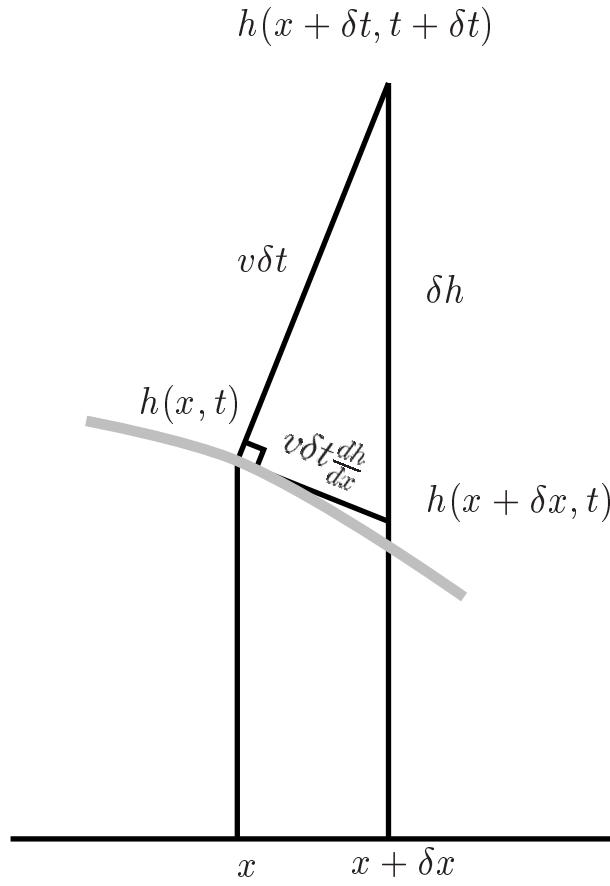


Fig. 1. Geometry for interface growth from a substrate. The interface is shown as the thick grey line. Here $(\delta h)^2 = (v\delta t)^2 \left[1 + \left(\frac{dh}{dx} \right)^2 \right]$.

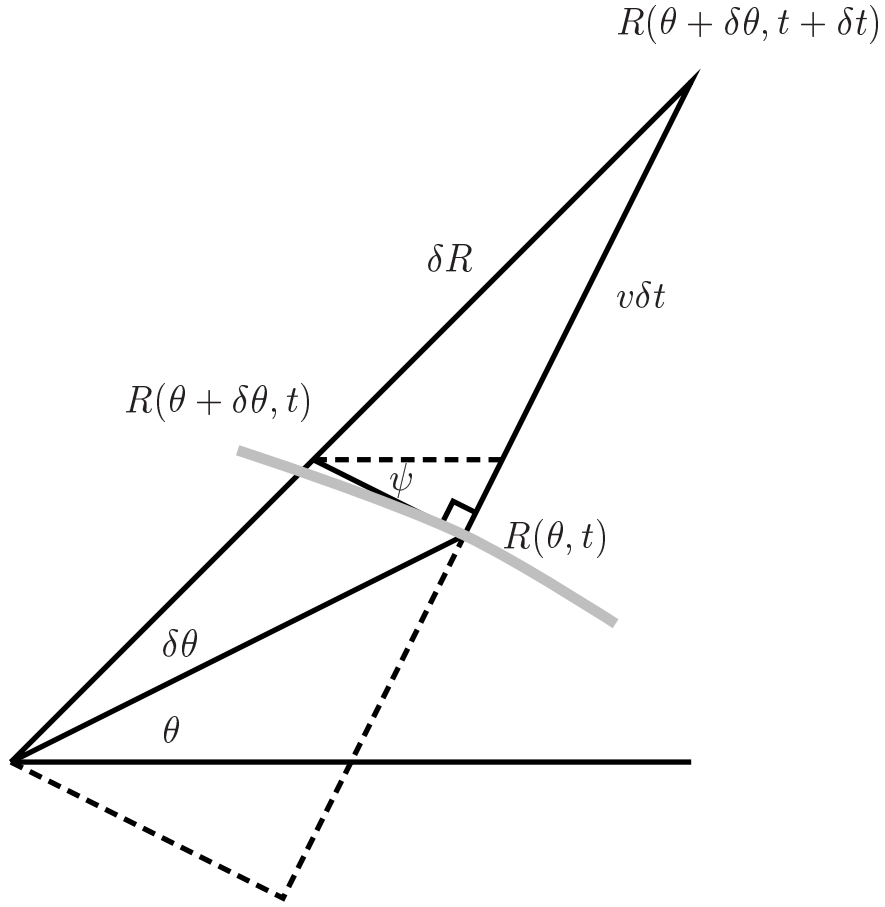


Fig. 2. Geometry for radial interface growth from a seed. The interface is shown as the thick grey line. Here $(\delta R)^2 = \frac{(v\delta t)^2 [R^2 + (v\delta t)^2 + 2Rv\delta t \sin(\theta + \psi)]}{[R \sin(\theta + \psi) + v\delta t]^2}$.

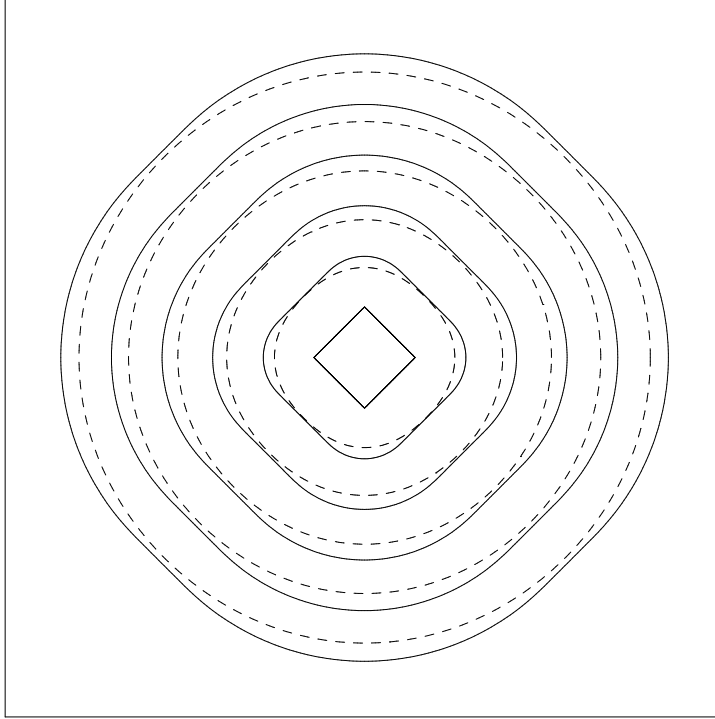


Fig. 3. Time snapshots of the evolution of an initial diamond shaped interface in the deterministic limit of the continuum radial growth equation. The solid line shows growth without surface tension and the dashed line is growth with surface tension.

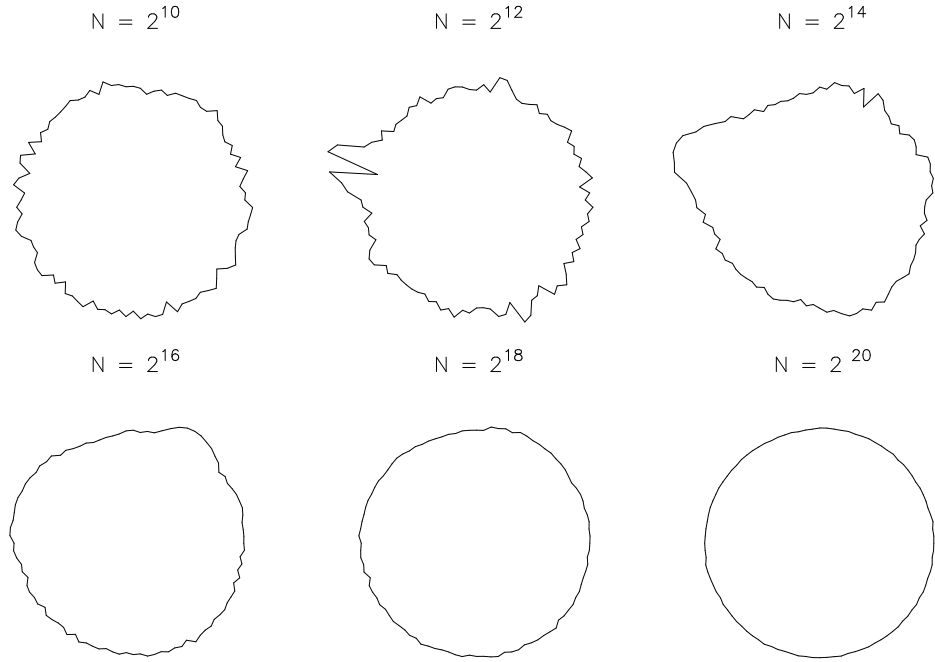


Fig. 4. Time snapshots of the evolution of an initial circular interface in the stochastic continuum radial growth equation. N is the number of iterations with time step $1/1000$.

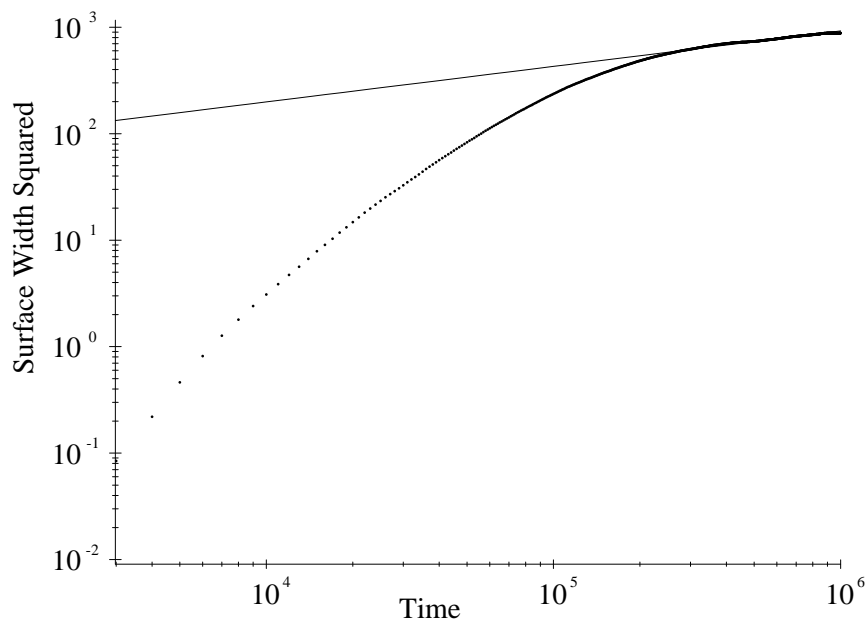


Fig. 5. Log-log plot of the surface width squared versus time (number of iterations) in the stochastic continuum radial growth equation. The solid line is the best fit line over the domain $t \in (2.5 \times 10^5, 10^6)$ with slope $\beta = 1/3$.